Seminar Handout (Week 12)

Some Inequalities

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1. Definitions

$$\eta_{(n+1)\times(n+1)} := \begin{bmatrix} \eta_{00} & \cdots & \eta_{0n} \\ \vdots & & \vdots \\ \eta_{n0} & \cdots & \eta_{nn} \end{bmatrix} := \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\Box := -\sum_{a,b=0}^{n} \eta_{ab}\partial_a\partial_b = \partial_t^2 - \Delta \ (\partial_0 := -\partial_t)$$

$$\Omega_{ab} := x_a\partial_b - x_b\partial_a \ \ (a,b = 0,1,\cdots,n),\ x_0 := t$$

$$L_i := \Omega_{0i}\ (i = 1,\cdots,n), \qquad L_0 := \sum_{a,b=0}^{n} \eta_{ab}x_a\partial_b$$

2. Lemmas

Lemma 4.9. Let $m \in \mathbb{N}$, then there is a constant $c = c(m,n) > 0$ s.t. for all $f, g \in W^{m,2} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant m$,
(i) $\|\nabla^\alpha(fg)\|_2 \leqslant c(\|f\|_\infty\|\nabla^m g\|_2 + \|\nabla^m f\|_2\|g\|_\infty)$
(ii) $\|\nabla^\alpha(fg) - f\nabla^\alpha g\|_2 \leqslant c(\|\nabla f\|_\infty\|\nabla^{m-1} g\|_2 + \|\nabla^m f\|_2\|g\|_\infty)$

Lemma 4.10. Let $n = 3$. Then the operators $\Omega_{ij}, 1 \leqslant i < j \leqslant 3$, are given in polar coordinates by

$$\Omega_{12} = \partial_\varphi$$
$$\Omega_{13} = -\cos\varphi\partial_\theta + \cot\theta\sin\varphi\partial_\varphi$$
$$\Omega_{23} = -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi$$
$$\partial_\theta = -\sin\varphi\Omega_{23} - \cos\varphi\Omega_{13}$$

Lemma 4.11.

$$[\Omega_{ab}, \Box] = 0, \quad a,b = 0,1,\cdots,n$$

Lemma 4.12. Let $a,b,c,d \in \{0,1,\cdots,n\}$.
(i) $[L_0, \Box] = -2\Box$, (ii) $[L_0, \Omega_{ab}] = 0$, (iii) $[L_0, \partial_a] = -\partial_a$,
(iv) $[\Omega_{ab}, \Omega_{cd}] = \eta_{bc}\Omega_{ad} + \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd}$, (v) $[\Omega_{ab}, \partial_c] = \eta_{bc}\partial_a - \eta_{ac}\partial_b$.

References.

[1] R. Racke, *Lectures on Nonlinear Evolution Equations.*